19 The Coulomb Phase of $SO(N)$

19.1 Monodromies of the Coupling

At a generic point in the classical moduli space $SO(N)$ with $N-2$ flavors is broken to $SO(2) \sim U(1)$. The effective holomorphic coupling

$$\tau = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2} \tag{19.1}$$

of the $U(1)$ gauge theory transforms under electric-magnetic duality as:

$$S : \tau \to -\frac{1}{\tau} \tag{19.2}$$

This is not a symmetry of the theory, it exchanges two equivalent descriptions. Shifting θ_{YM} by 2π is a symmetry

$$T : \tau \to \tau + 1 \tag{19.3}$$

(S and T can be thought of as the generators of $SL(2, Z)$.) For large $z = \det M$ we know that

$$\tau \approx \frac{i}{2\pi} \ln\left(\frac{z}{\Lambda^b}\right) \tag{19.4}$$

We see that τ has a singluarity in the complex variable z at $z = \infty$. Consider moving the VEVs around this point

$$\Phi_j \to e^{2\pi i}\Phi_j \tag{19.5}$$

which yields

$$z \to e^{2F2\pi i}z = e^{b2\pi i}z \tag{19.6}$$

or

$$\tau \to \tau - b \tag{19.7}$$

Such a transformation when going around a complex singularity is called a monodromy. Thus the monodromy at ∞ of τ is

$$\mathcal{M}_\infty = T^{-b} \tag{19.8}$$

So we see that τ is not single-valued, even at weak coupling. However

$$\frac{4\pi}{g^2} = \mathrm{Im}\tau \tag{19.9}$$

is invariant under $\mathcal{M}_\infty$ (i.e. is single-valued) at weak coupling. If $\mathrm{Im}\tau$ was single-valued everywhere then it's derivatives would be well defined and (since τ is holomorphic) we have

$$(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})\mathrm{Im}\tau = 0 \tag{19.10}$$

where $z = x + iy$. So $\mathrm{Im}\tau$ is a harmonic function, which cannot be positive definite, so g would be imaginary at somewhere in the moduli space. So we conclude that $\mathrm{Im}\tau$ is not single-valued everywhere.

This means there are two possibilities, the moduli space has some complicated topology or there are singular points z_i. Singular points have a physical interpretation as particles becoming massless at that point. We will see that this does happen in this theory. The singular points have monodromies, at least two of which don't commute with $\mathcal{M}_\infty$, otherwise τ would be single-valued.

19.2 Flowing from $F = N - 1$ to $F = N - 2$

The dual of $SO(N)$ with $N - 1$ flavors is (for $N > 3$)

	$SO(3)$	$SU(N-1)$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\frac{N-2}{N-1}$
M	$\mathbf{1}$	$\square\square$	$\frac{2}{N-1}$

with

$$W = \frac{M_{ji}}{2\mu}\phi^j\phi^i - \frac{1}{64\Lambda_{N,N-1}^{2N-5}}\det M \tag{19.11}$$

Integrate out one flavor by adding $\frac{1}{2}mM_{N-1,N-1}$ and The equations of motion give

$$\phi^{N-1}\phi^{N-1} = \frac{\mu\det M'}{32\Lambda_{N,N-1}^{2N-5}} - \mu m \tag{19.12}$$

which (near $\det M' = 0$) breaks $SO(3)$ to $U(1)$. There are corrections from instantons, so the effective superpotential is:

$$W_{\text{eff}} = \frac{1}{2\mu} f\left(\frac{\det M'}{\Lambda_{N,N-2}^{2N-4}}\right) M'_{ij} \phi^{+i} \phi^{-j} \qquad (19.13)$$

Note that large $\det M'$ is at strong coupling.

If $r = \text{rank}(M')$, then there are $F - r = N - 2 - r$ massless flavors (q^+ and q^-) at $\det M' = 0$. Consider ϕ_0^i such that $\det M'_0 = 0$, and take

$$\phi^i - \phi_0^i \rightarrow e^{2\pi i}(\phi^i - \phi_0^i) \qquad (19.14)$$

then

$$z = \det M' \rightarrow \exp\left(2(F - r)2\pi i\right) z \qquad (19.15)$$

and

$$\widetilde{\tau} = -\frac{i}{2\pi} \ln\left(\det M'\right) + c \qquad (19.16)$$

so

$$\widetilde{\tau} \rightarrow \widetilde{\tau} + F - r \qquad (19.17)$$

Defining the duality transformation D by

$$\widetilde{\tau} = D\tau \qquad (19.18)$$

we have

$$\mathcal{M}_0 = D^{-1}T^{F-r}D \qquad (19.19)$$

Since we require

$$[\mathcal{M}_0, \mathcal{M}_\infty] \neq 0 \qquad (19.20)$$

we must have

$$D \propto S^{2n+1} \qquad (19.21)$$

So we see that $\phi^\pm$ are monopoles and that $\widetilde{\tau} \rightarrow 0$ implies $\tau \rightarrow \infty$.

19.3 Diversion on $N = 3$

Recall the special case of $SO(N)$ with $N-1$ flavors, this was dual to $SO(3)$ with $N-1$ flavors with a special superpotential:

$$W = \frac{M_{ji}}{2\mu}\phi^j\phi^i - \frac{1}{64\Lambda_{N,N-1}^{2N-5}}\det M \tag{19.22}$$

By considering the dual of this dual we can see that this implies that $N = 3$ is a special case. To get the correct dual of the dual, the dual of $SO(3)$ with F flavors must have an extra term as well. That is the $SO(F+1)$ dual must have a superpotential

$$\widetilde{W} = \frac{M_{ji}}{2\mu}\phi^j\phi^i + \epsilon\alpha\det(\phi^j\phi^i) \tag{19.23}$$

The value of α will be determined by demanding consistency. The factor ϵ reflects the fact that $SO(3)$ has a discrete axial Z_{4F} symmetry

$$Q \to \exp\left(\frac{2\pi i}{4F}\right)Q \tag{19.24}$$

while $SO(F+1)$ only has a Z_{2F} symmetry (for $F > 2$). Under the full Z_{4F} the $\det(\phi^j\phi^i)$ term changes sign, and

$$\theta_{\text{YM}} \to \theta_{\text{YM}} + \pi \tag{19.25}$$

Using (19.23) we find that the dual of the dual of $SO(N)$ with $N-1$ flavors has a superpotential

$$\widetilde{\widetilde{W}} = \frac{MN}{2\mu} + \frac{N^{ij}}{2\mu}d_jd_i - \frac{\det M}{64\Lambda_{N,N-1}^{2N-5}} + \epsilon\alpha\det(d_jd_i) \tag{19.26}$$

With $\widetilde{\mu} = -\mu$, the N equation of motion sets $M_{ji} = d_jd_i$ as we expect, and for $\epsilon = 1$, $\widetilde{\widetilde{W}} = 0$ if

$$\alpha = \frac{1}{64\Lambda_{N,N-1}^{2N-5}} \tag{19.27}$$

So the dual of the dual is the original theory for $\epsilon = 1$.

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19.4 The Dyonic Dual

To determine what happens for VEVs of the order of the strong interaction scale, we make use of the the second dual of $SO(N)$ with $N-1$ flavors which has $\epsilon = -1$. For reasons that will become apparent, we will refer to this as the dyonic dual, and the usual dual as the magnetic dual.

The dyonic dual is

	$SO(N)$	$SU(F = N - 1)$	$U(1)_R$
d	$\square$	$\square$	$\frac{1}{F}$

with

$$W_{\text{dyonic}} = -\frac{\det(d_i d_j)}{32\Lambda_{N,N-1}^{2N-5}} \tag{19.28}$$

Integrating out a flavor we have

$$W_{\text{dyonic}} = -\frac{\det(d_i d_j)}{32\Lambda_{N,N-1}^{2N-5}} + \frac{1}{2}m d_F d_F \tag{19.29}$$

where we use the notation

$$d = (d_i, d_F), \ i = 1, \ldots, N-2 \tag{19.30}$$

The equation of motion for d_i gives $d_i d_F = 0$. For $\det(d_i d_j) \neq 0$, $SO(N)$ is broken to $U(1)$ and we have

$$W_{\text{eff}} = \frac{1}{2}m\left(1 - \frac{\det(d_i d_j)}{16\Lambda_{N,N-1}^{2N-5}}\right)d_F^+ d_F^- \tag{19.31}$$

Near

$$\det(d_i d_j) = 16\Lambda_{N,N-1}^{2N-5} \tag{19.32}$$

d_F^+ and d_F^- are light. Since they are duals of monopoles with $\theta_{\text{YM}} \to \theta_{\text{YM}} + \pi$, they are dyons. The charges are such that

$$q^{\pm i}Q_i \sim d_F^{\pm} \tag{19.33}$$

Taking $m \to 0$, the dyon point moves towards the monopole point, when $m = 0$ we know that the theory has an $SO(3)$.

So we have a web of three dualities that each describes a region of the moduli space where different degrees of freedom become light. Integrating out a flavor in the electric theory gives $SO(N)$ with $N-3$ flavors which has two branches, one with a runaway vacuum and the other with a moduli space exhibiting confinement. Adding the corresponding linear term in the magnetic dual gives a VEV to one flavor of monopoles throught the dual Meissner effect this corresponds to confinement in the electric theory. The remaining light monopoles can be identified with the hybrids $h^i = W_\alpha W^\alpha Q^{N-4}$. Adding a mass term $\frac{1}{2}m_2 d_{N-2} d_{N-2}$ in the dyonic dual gives a VEV

$$\langle d_F^+ d_F^- \rangle = \frac{m_2 16 \Lambda_{N,N-2}^{2N-4}}{m \det M'} \tag{19.34}$$

which gives an effective superpotential

$$W_{\text{eff}} = \frac{m_2 8 \Lambda_{N,N-2}^{2N-4}}{\det M'} \tag{19.35}$$

This theory displays essentially the same type of physics ans the $\mathcal{N} = 2$ Seiberg-Witten models, and there are several consistency checks between them.

19.5   Phases of Gauge Theories

A gauge invariant order parameter to distinguish between different phases of a gauge theory is provided by the Wilson Loop:

$$W(C) = \text{Tr} P e^{i \int_C A_\mu dx^\mu} \tag{19.36}$$

If C is a rectangle with sides T and R then for large T we can extract the potential between two infinitely heavy charged sources:

$$\langle W(C) \rangle = e^{-TV(R)} \tag{19.37}$$

The various cases for the phases are

$$
\begin{aligned}
&\text{Coulomb (or fixed point)}: &&V(R) \sim \tfrac{1}{R} \\
&\text{free electric}: &&V(R) \sim \tfrac{1}{R \ln(R\Lambda)} \\
&\text{free magnetic}: &&V(R) \sim \tfrac{\ln(R\Lambda)}{R} \\
&\text{Higgs}: &&V(R) \sim const \\
&\text{confining}: &&V(R) \sim \sigma R
\end{aligned}
$$

Replacing the electric charges in the loop with monopoles, we have an 't Hooft loop. Because of electric-magnetic duality, this interchanges free electic and free magnetic and interchanges Higgs and confinement. We have seen in some detail how this works in the Coulomb phase of $SO(N)$, and it is very suggestive of how $\mathcal{N} = 1$ dualities work in general.

References

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